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Integrable and Solvable Systems, and Relations among them [and Discussion]

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Integrable and solvable systems, and relations among them

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There are several different classes of differential equations that may be described as 'integrable' or 'solvable'. For example, there are completely integrable dynamical systems; equations such as the sine–Gordon equation, which admit soliton solutions; and the self-dual gauge-field equations in four dimensions (with generalizations in arbitrarily large dimension). This lecture discusses two ideas that link all of these together: one is the Painlevé property, which says (roughly speaking) that all solutions to the equations are meromorphic; the other is that many of the equations are special cases (i.e. reductions) of others.

1. Introduction

The papers at this Discussion Meeting have dealt with differential equations that may be described as integrable or solvable. These include, in particular, equations admitting soliton solutions, and integrable dynamical systems. One thread that links them together is, as we have heard, the involvement of (infinite-dimensional) Kac–Moody algebras; another is the 'Painlevé property', which says (roughly speaking) that the solutions to the equations are all meromorphic (this is discussed in more detail in §2). A third linking thread involves the self-dual gauge-field equations in four dimensions, which are 'completely solvable'. The link is provided by the fact that many (and perhaps all?) of the ordinary or partial differential equations that are regarded as being integrable or solvable may be obtained from the self-duality equations (or its generalizations) by reduction. In a sense, they are special cases of the self-duality equations. Section 3 of this paper is devoted to providing some evidence for this idea.

By the term 'reduction', I mean the following: suppose we have a set of coupled partial differential equations (n equations in n unknowns) with m independent variables. Then we reduce either by reducing n, or reducing m, or both. By way of example, consider the following case, where $m = n = 2$:

$$\Box\phi = e^{\phi} - e^{-\psi},$$
$$\Box\psi = e^{\psi} - e^{-\phi},$$

with $\Box = \partial_t^2 - \partial_x^2$ being the wave operator in two dimensions. Because these equations are invariant under translations $x \mapsto x + k$, we can reduce by requiring ϕ and ψ to be functions of t only, and so obtain the system

$$\phi'' = e^{\phi} - e^{-\psi},$$
$$\psi'' = e^{\psi} - e^{-\phi}.$$

Alternatively, because the equations are invariant under the interchange of ϕ and ψ, we may reduce by setting $\psi = \phi$, and obtain the sinh–Gordon equation

$$\Box\phi = 2 \sinh \phi.$$

The main point is that in each case, it is the presence of a symmetry that enables the reduction to be made.

2. Integrability and solvability

We are interested in certain special systems of nonlinear ordinary or partial differential equations: special in the sense of being integrable or solvable. The precise definition of what these terms mean is a little vague. For systems of ordinary differential equations (o.d.es) one can define integrability in terms of the existence of enough 'constants of the motion'. A more precise (and stronger) idea is that of 'algebraic complete integrability', which requires that the equations can be integrated in terms of Abelian functions, and that (almost) every solution corresponds to linear flow on an Abelian variety (see the paper by Professor P. van Moerbeke). For systems of partial differential equations (p.d.es), the situation is somewhat more obscure. There are equations such as the K.d.V. equation, which admit an infinite number of local conserved currents, and which may justifiably be referred to as 'completely integrable Hamiltonian systems'. But there are also systems such as the self-dual gauge-field equations, which appear not to fall into such a category, yet are just as 'solvable' as the soliton-type equations. Equally, it is unsatisfactory to define solvability in terms of one particular solution technique, such as the inverse scattering transform, since such a technique has only limited applicability, and does not cover all the systems that one would wish to describe as being solvable or integrable.

One property that integrable or solvable p.d.es *do* appear to have in common is that they can be expressed as consistency conditions for the solution of overdetermined systems of *linear* p.d.es. Indeed, this fact underlies both the 'inverse scattering' technique for solving soliton-type equations, and the 'twistor' technique for solving the self-dual gauge-field and related equations. But the fact that a system of p.d.es can be written as a consistency condition, does not automatically guarantee that that system is solvable: there are several examples of equations that are expressible as consistency conditions, but which are believed *not* to be integrable. It seems that for the consistency conditions of an overdetermined system of linear equations to be integrable, one needs that linear system to be of a certain type; but just what the 'certain type' is, has not yet, to my knowledge, been elucidated.

A useful test for integrability is that of S. Kowalewski, who applied it to autonomous systems of o.d.es (dynamical systems). The criterion is the 'Painlevé property', namely that the solutions of the equations, as functions on $\mathbb{C}$ (i.e. functions of complex time), should have no singularities other than poles (i.e. that they should be meromorphic). This has proved to be an effective test for integrability, and, more specifically, for algebraic complete integrability. The idea (conjecture) is that the Painlevé property implies integrability. The reverse implication does not hold in general; one reason for this is that the Painlevé property is not invariant under a change of variables, whereas integrability (depending on how one defines it) sometimes is. A trivial example is that of a single o.d.e. $x' = f(x)$, which is 'integrable by quadrature', but does not, in general, have the Painlevé property. On the other hand, the more exacting requirement of algebraic complete integrability implies the Painlevé property (and seems, judging by the cases that have been investigated so far, to be implied by it).

Various suggestions have recently been made on how to extend the applicability of this test from o.d.es to p.d.es. One idea is to require that whenever a p.d.e. has symmetries that allow it to be reduced to an o.d.e., then that o.d.e. should possess the Painlevé property. It appears to be the case that p.d.es which are solvable by the inverse scattering transform pass this test, whereas several equations that are believed not to be integrable fail it. But this test seems unsatisfactory, in that most p.d.es do not have any symmetries at all, and therefore cannot be reduced to o.d.es.

A better idea is as follows. Suppose that there are d independent variables, and that the system of p.d.es in question has coefficients that are holomorphic on $\mathbb{C}^d$. We cannot simply require that all the solutions of this system be meromorphic on $\mathbb{C}^d$, since arbitrarily nasty singularities can occur along characteristic hypersurfaces (even for the most benign equation, such as Laplace's equation). But the following formulation of the Painlevé property avoids this problem.

Painlevé property 1. (P1) *If* S *is a holomorphic non-characteristic hypersurface in* $\mathbb{C}^{\mathrm{d}}$, *then every solution that is holomorphic on* $\mathbb{C}^{\mathrm{d}}\backslash \mathrm{S}$ *extends to a meromorphic solution on* $\mathbb{C}^{\mathrm{d}}$.

In other words, if a solution has a singularity on a non-characteristic hypersurface, then that singularity is a pole and nothing worse. For example, the wave equation $\phi_{tt}-\phi_{xx} = 0$ in two dimensions satisfies P1, as one sees by considering its general solution $\phi(x, t) = f(x-t)+g(x+t)$. In fact it clearly satisfies a stronger version of P1, in which the word 'meromorphic' is replaced by 'holomorphic'.

A slightly weaker form of the Painlevé property was formulated a couple of years ago by Weiss *et al.* (1983). It involves looking for solutions ϕ of the system of p.d.es in the form

$$\phi = \sigma^{-\alpha} \sum_{n=0}^{\infty} \phi_n \sigma^n,$$

where σ is a holomorphic function whose vanishing defines a non-characteristic hypersurface. Substituting this series into the p.d.es yields conditions on the number α and recursion relations for the functions ϕ_n. The requirement (let us refer to it as P2) is that α should turn out to be a non-negative integer, and the recursion relations should be consistent, and that the series expansion should contain the correct number of arbitrary functions (counting σ as one of them).

In all the cases that have been checked so far, it has been found that integrable equations satisfy P2 (perhaps after a change of variables), whereas non-integrable equations fail it. To establish P1 is more difficult (P1 implies P2, but the reverse implication need not hold in general). However, it seems that in practice, P2 is sufficient to ensure integrability.

In summary, then, the Painlevé property (in either of the forms P1 or P2) seems to be a a useful indicator of integrability or solvability, or both. It is worth also pointing out that the Painlevé property is preserved under reductions of the kind discussed in §1.

3. The self-duality equations, and reductions

I now want to discuss the self-dual gauge-field equations on Euclidean 4-space $\mathbb{R}^4$. Very briefly, the arrangement is as follows. Let G be a Lie group (the 'gauge group') and $\mathfrak{g}$ its Lie algebra. A gauge potential (connection) A is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}^4$. The corresponding gauge field (curvature) is the $\mathfrak{g}$-valued 2-form $F = \mathrm{d}A+[A, A]$. Two gauge potentials A and A' are regarded as being equivalent if they are related by a gauge transformation

$$A' = \Omega^{-1} A\Omega + \Omega^{-1}\,\mathrm{d}\Omega,$$

where Ω is a G-valued function on $\mathbb{R}^4$. The self-duality equations are $*F = F$, where $*$ is the

Hodge duality operator, which maps 2-forms on $\mathbb{R}^4$ to 2-forms. In terms of the standard coordinates $x^a = (x^0, x^1, x^2, x^3)$ on $\mathbb{R}^4$, the gauge field $F_{ab} = -F_{ba}$ satisfies

$$F_{01} = F_{23}, F_{02} = -F_{13}, F_{03} = F_{12}.$$

These form a set of coupled first-order nonlinear p.d.es for A_a. They are underdetermined (fewer equations than unknowns), but this underdeterminacy can be removed by imposing a 'gauge condition' such as $A_0 = 0$. (The self-duality equations are invariant under gauge transformations.)

The self-duality equations are completely solvable as a consequence of the 'twistor' correspondence, which relates solutions of the equations to certain holomorphic vector bundles, and leads to various different ways of constructing solutions A (Atiyah 1979; Ward 1981). The equations possess the 'strong' Painlevé property P1, although one has to reformulate P1 slightly so as to take account of the gauge invariance. Namely, the following is true. Let S be a non-characteristic holomorphic hypersurface in $\mathbb{C}^4$, and A a self-dual gauge potential that is holomorphic on $\mathbb{C}^4 \backslash S$. Let p be any point on S. Then there is a neighbourhood W of p in $\mathbb{C}^4$ such that A, possibly after a gauge transformation, is meromorphic on W (Ward 1984*a*).

I now want to illustrate the fact that many 'nice' equations may be derived from the self-duality equations by reduction. The first example is obtained by reducing from $\mathbb{R}^4$ to $\mathbb{R}^3$: we do this by assuming the gauge potential A_a to be independent of x^0 (say). Then writing $\Phi = A_0$ and regarding the remaining components (A_1, A_2, A_3) as defining a G-connection on $\mathbb{R}^3$, the self-duality equations reduce to

$$\mathrm{D}\Phi = \mathrm{D} \times A,$$

where $\mathrm{D}\Phi = \mathrm{d}\Phi + [A, \Phi]$ is the covariant gradient of Φ, and $\mathrm{D} \times A$ the covariant curl of A (the dual of the curvature form). These are known as the Bogomolny equations, and are relevant to static magnetic poles (see the paper by Sir Michael Atiyah (this symposium)).

My next examples involve reduction to two dimensions ($\mathbb{R}^2$). Leznov and Saveliev (1980) showed that one such reduction produces the Toda field equations

$$\Box \phi_\alpha = \exp \sum_\beta K_{\alpha\beta} \phi_\beta,$$

where $\Box$ is the wave operator on $\mathbb{R}^2$, α and β label simple roots of the Lie algebra $\mathfrak{g}$, and $K_{\alpha\beta}$ is the Cartan matrix of $\mathfrak{g}$. Reducing by one more dimension yields the 'Toda molecule', an integrable dynamical system.

Let me exhibit another two-dimensional example, one that does not fall within the above scheme. Let the gauge group G be $\mathrm{SL}(2, \mathbb{C})$, so that the A_a are 2×2 matrices, and suppose them to be functions of x^0 and x^1 only. Then reduce the number of dependent variables by taking A_0 and A_1 to be diagonal, while A_2 and A_3 have the form

$$A_2 \pm \mathrm{i}A_3 = \tfrac{1}{2}\mathrm{i}\begin{bmatrix} 0 & \exp(\mp\tfrac{1}{2}\mathrm{i}\phi) \\ \exp(\pm\tfrac{1}{2}\mathrm{i}\phi) & 0 \end{bmatrix}$$

(ϕ being a scalar function of x^0 and x^1). This reduction is consistent with the self-duality equations, which reduce to

$$\Delta\phi = \sin\phi$$

(Δ being the Laplacian on $\mathbb{R}^2$), together with equations on A_0 and A_1 which need not concern us. In other words, the self-duality equations reduce to the elliptic sine–Gordon equation. (If we had begun in Minkowski space–time rather than Euclidean 4-space, we would have ended up with the usual hyperbolic sine–Gordon equation).

A comprehensive analysis of reductions of the above kind has not, as yet, been made. The possibility exists that many other soliton equations, in two or more dimensions, may turn out to be special cases of the self-duality equations; but this remains to be seen.

A different family of reductions arises if we rewrite the self-duality equations in a different form (due to C. N. Yang). In terms of the four real coordinates x^a, define two complex coordinates $y = x^0 + \mathrm{i}x^1$ and $z = x^2 + \mathrm{i}x^3$. If A_a is a self-dual gauge potential, then there exist g-valued scalar fields H and K such that

$$A_y = H^{-1}\partial_y H, \quad A_z = H^{-1}\partial_z H,$$
$$A_{\bar{y}} = K^{-1}\partial_{\bar{y}} K, \quad A_{\bar{z}} = K^{-1}\partial_{\bar{z}} K,$$

where ∂_y denotes ∂/∂_y and so forth. The integrability condition for these expressions forms part of the self-duality equations. The remainder of the self-duality equations is as follows: put $J = HK^{-1}$, and then

$$\partial_{\bar{y}}(J^{-1}\partial_y J) + \partial_{\bar{z}}(J^{-1}\partial_z J) = 0. \tag{1}$$

So we obtain a G-valued scalar field J satisfying (1); conversely, a solution J of (1) determines a unique self-dual gauge field (unique, that is, up to gauge transformations; for the field J is gauge-invariant). So (1) is a neat form of the self-duality equations; its main defects are that SO(4)-invariance, which was present in the original formulation, has been lost; and that the geometrical interpretation (in terms of connections and curvatures) has also been lost.

It is worth remarking that for the group $G = \mathrm{SU}(2)$, Jimbo *et al.* (1982) have shown directly that (1) satisfies the Painlevé property P2.

Equation (1) has many interesting reductions: one is obtained as follows. Require J to depend only on $\xi = \mathrm{Im}(y)$ and $\rho = 2|z|$ (i.e., factor out a translation and a rotation, both of which are symmetries of (1)). Equation (1) then becomes

$$\partial_\xi(J^{-1}\partial_\xi J) + \partial_\rho(J^{-1}\partial_\rho J) + \rho^{-1}J^{-1}\partial_\rho J = 0. \tag{2}$$

Then let the gauge group be SU(2), so that J takes values in SU(2), and impose the further constraint that each entry in the 2×2 matrix J be real-valued. With this constraint, (2) is (a form of) the 'Ernst equation' of general relativity, the solutions of which correspond to stationary axisymmetric solutions of Einstein's vacuum equations.

Another reduction of (1) is obtained by factoring out two translations, namely by requiring J to be a function of x^0 and x^2 only. This gives

$$\partial_\mu(J^{-1}\partial_\mu J) = 0, \tag{3}$$

where the index μ takes on the two values 0 and 2, and the Einstein summation convention operates. Equation (3) is the well known 'chiral field equation' in $\mathbb{R}^2$. It, in turn, reduces to other equations much studied by theoretical physicists, such as the equations of the CP^n model: if we require that J satisfy $J^2 = 1$, so that it can be written as $J = 1 - 2P$ with $P^2 = P$, then (3) becomes $[P, \Delta P] = 0$, which is a form of the CP^n equations (Din *et al.* 1984).

There is one further family of reductions of the self-duality equations that I want to discuss, and this involves reducing all the way down to one dimension, i.e. to o.d.es. Suppose that the gauge potential A_a is a function only of x^0, and choose a gauge in which $A_0 = 0$. The self-duality equations reduce to

$$A_i' = \tfrac{1}{2}\epsilon_{ijk}[A_j, A_k], \tag{4}$$

where $'$ denotes $\mathrm{d}/\mathrm{d}x^0$, the indices i, j, k run over 1, 2, 3, and ϵ_{ijk} is the totally skew tensor with $\epsilon_{123} = 1$. Think of i, j, k as indices in the Lie algebra so(3); then A takes values in $\mathscr{A} = \mathrm{so}(3) \otimes \mathfrak{g}$, the tensor product of the two Lie algebras so(3) and $\mathfrak{g}$ (considered as vector spaces). The equations (4) are known as the Nahm equations.

Note that $\mathscr{A}$ has the structure of an algebra, with the multiplication operation being the natural one induced by the multiplications on the two Lie algebras. Denote the product of $A, B \in \mathscr{A}$ by $(A, B) \in \mathscr{A}$. The algebra $\mathscr{A}$ is commutative, but not, in general, associative. The equations (4) are simply the natural flow on this algebra: namely, $A: \mathbb{R} \rightarrow \mathscr{A}$ satisfies $A' = (A, A)$.

Reduction of the number of dependent variables is effected by simply restricting to a subalgebra $\mathscr{H}$ of $\mathscr{A}$. For example, take $\mathfrak{g} = \mathrm{so}(3)$, and let $\mathscr{H}$ be the 'diagonal' in $\mathrm{so}(3) \otimes \mathrm{so}(3)$. In other words, if so(3) is generated by σ_i with $[\sigma_1, \sigma_2] = \sigma_3$ etc., then $\mathscr{H}$ is generated by $\{\sigma_1 \otimes \sigma_1, \sigma_2 \otimes \sigma_2, \sigma_3 \otimes \sigma_3\}$. The flow on this three-dimensional algebra corresponds to the system

$$f' = 2gh, \quad g' = 2fh, \quad h' = 2fg,$$

which (after rescaling) are Euler's equations for a spinning top, a classical example of an algebraically completely integrable system.

I conjecture that every algebra $\mathscr{H}$ of the type described above (i.e. a subalgebra of $\mathrm{so}(3) \otimes \mathfrak{g}$ for some Lie algebra $\mathfrak{g}$) gives rise to an integrable system of o.d.es. It may also transpire that a quadratic system of o.d.es (i.e. $f_\alpha' = Q_\alpha^{\beta\gamma} f_\beta f_\gamma$ with $Q_\alpha^{\beta\gamma}$ constant) is completely integrable *only* if it corresponds to the flow on such an algebra $\mathscr{H}$. For example, preliminary investigation seems to indicate that the integrable cases of geodesic flow on SO(n) fall within this framework.

4. Conclusions

The main point of this lecture has been to emphasize the commanding position occupied by the self-dual gauge-field equations among integrable or solvable systems. It seems appropriate to look for common features among the plethora of such systems; and the relation between two systems of one being a reduction of the other, is the most obvious relation to look for.

Let me mention some variants and generalizations of the self-duality equations. First, there are the self-dual Einstein equations, which refer to four-dimensional Riemannian spaces, with self-dual conformal curvature, and satisfying the Einstein condition $R_{ab} = \lambda g_{ab}$. These are completely solvable in the same sense as the self-dual gauge-field equations (Penrose 1976; Ward 1980). Also completely solvable are a host of higher-dimensional (i.e. dim > 4) generalizations of the self-duality equations, many of which reduce to the latter equations. All the possibilities in this regard have not yet been fully analysed, but a preliminary account is given in Ward (1984*b*).

I shall end by listing some questions that seem to me to be interesting. First, is there a clear definition of 'integrable–solvable' that includes all the examples I have mentioned? Two possible approaches to such a definition might be the associated linear system, and the Painlevé property. Secondly, how many 'master' integrable–solvable equations are there, in the sense that all integrable–solvable equations can be derived from these by reduction? It is possible

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that there may, in fact, be very few such basic equations. Thirdly, is there a connection between integrability and supersymmetry? It seems to be the case that solvable field theories admit supersymmetric extensions; is this a general phenomenon?

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Discussion

M. TABOR (*Department of Applied Physics, Columbia University, New York, U.S.A.*). Could Dr Ward define more precisely the term 'non-characteristic hypersurface' and indicate how his version of 'Painlevé property' might be proved for, say, members of the Kadomtzev–Petviashvili hierarchy; for example, the K.d.V. equation?

R. S. WARD. My interest is mainly in relativistic equations (i.e. equations invariant under the Poincaré group in $d+1$ dimensions, for any d), such as sine–Gordon, Yang–Mills, and so forth; in such cases, the characteristic hypersurfaces are simply the null hypersurfaces. On the other hand, for the K.d.V. equation

$$U_t + U U_x + U_{xxx} = 0,$$

the characteristic lines are $t = $ constant (these are, for example, implicitly excluded by Weiss *et al.* in their definition of P2).

As to how P1 could be proved for, say, the K.d.V. equation, the answer is that I do not know: it might be very difficult. One 'almost' has a proof for the sine–Gordon equation, since it is a reduction of the self-duality equations. (I have to say 'almost' because P1 only holds for the self-duality equations *in certain gauges*, and it is not clear *a priori* that the gauge one chooses when reducing to the sine–Gordon equation is one of these.) Perhaps something similar might work for the K.d.V. equation, if one could see how to obtain it from the self-duality equations.

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